Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 16, 2004*
Accepted: *September 10, 2004*

On spin chains and field theories

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Abstract: We point out that the existence of global symmetries in a field theory is not an essential ingredient in its relation with an integrable model. We describe an obvious construction which, given an integrable spin chain, yields a field theory whose 1-loop scale transformations are generated by the spin chain hamiltonian. We also identify a necessary condition for a given field theory to be related to an integrable spin chain.
As an example, we describe an anisotropic and parity-breaking generalization of the XXZ Heisenberg spin chain and its associated field theory. The system has no nonabelian global symmetries and generally does not admit a supersymmetric extension without the introduction of more propagating bosonic fields. For the case of a 2-state chain we find the spectrum and the eigenstates. For certain values of its coupling constants the field theory associated to this general type of chain is the bosonic sector of the q-deformation of $\mathcal{N}=4$ SYM theory.

Keywords: Renormalization Group, Duality in Gauge Field Theories, Lattice Integrable Models.

 http://jhep.sissa.it/archive/papers/jhep092004023/jhep092004023.pdf

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1. Introduction

The dilatation operator at 1-loop level in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) has been subject to extensive investigation over the past few years. A surprising result was that its action on gauge invariant operators can be mapped into the action of the hamiltonian of a spin chain on the vectors in its Hilbert space. The first steps in this direction were taken in [1] where it was shown that the action of the 1-loop dilatation operator on scalar operators can be described by an SO(6) spin chain in the fundamental representation. Other bosonic sectors of $\mathcal{N} = 4$ SYM enjoy similar properties (see e.g. [2]). Combining these results as well as some partial supersymmetric extensions [3] led to the realization that the dilatation operator for the full theory is described by an $\mathrm{PSU}(2,2|4)$ spin chain [4]. In this context, the continuum limit of some of the bosonic spin chains and their relation to a two-dimensional sigma model were discussed in [5].

An apparently different integrable structure was found in the world sheet theory of the bosonic string [6] and superstring [7] in $AdS_5 \times S^5$. Through the AdS/CFT correspondence, this translates into a certain algebraic structure in $\mathcal{N} = 4$ SYM which was shown [8] to be consistent with the $\mathrm{PSU}(2,2|4)$ spin chain representation of the dilatation operator.

More involved collections of operators in $\mathcal{N} = 4$ SYM also have certain relations with integrable models. The world sheet theory of strings in $AdS_5 \times S^5$ expanded around certain semiclassical solutions [9] has a description in terms of integrable models (see [10] and [11] for a review). Then, the AdS/CFT correspondence implies that the set of operators dual to excitations around this semiclassical configurations should exhibit a similar integrable structure.

The dilatation operator at higher loops can also be described as an operator acting on the spin chain appearing at the 1-loop level [1, 3]. Its interpretation as a deformation of the spin chain hamiltonian remains, however, an open question.

The relation between field theories and integrable models is not limited to very symmetric theories like $\mathcal{N} = 4$ SYM. For example, orbifolds of $\mathcal{N} = 4$ SYM in the planar limit were discussed in [12]. In the realm of non-supersymmetric theories, it was argued in [13] that in certain QCD processes the summation of Feynman graphs leads to an integrable model [14]. Following this line of reasoning, it was shown (see e.g. [15]) that various collections of operators are closed under scale transformations and their anomalous dimensions were determined.

Given this amount of evidence it is interesting to ask which 4-dimensional field theories can exhibit an integrable structure in the sense reviewed above. Some general observations immediately come to mind. For example, planar orbifolds of theories exhibiting an integrable structure will continue to have one due to the "inheritance principle" [16]. Indeed, it was shown that in such theories all planar diagrams are inherited from the parent theory, up to a trivial rescaling of the gauge coupling. Also, deformations by operators of dimension two do not affect the 1-loop scale transformations and thus preserve whatever integrable structure the undeformed theory might posses.

A complementary approach to the question formulated above asks which integrable models can describe the 1-loop dilatation operator of some field theory. If the field theory is in the planar limit, the choice is limited to models which have a lattice description with nearest neighbor interactions — a spin chain.

In this note we will take a small step in this second direction. Starting from a general integrable spin chain with only nearest neighbor interactions we show that its hamiltonian is the 1-loop dilatation operator for some sector of a family of 4-dimensional field theories. The general construction of the field theory associated to a given spin chain described in section 2 is straightforward and does not require the spin chain to have any symmetry properties. Depending on the detailed properties of the spin chain hamiltonian, it is sometimes possible that the resulting lagrangian is part of a supersymmetric theory. The construction implies a necessary condition for a given field theory to be related to an integrable spin chain and suggests a way to attempt to engineer a spin chain once a field theory is given.

We then proceed to analyze in detail an example. With the eventual goal of constructing a spin chain description of the q-deformation of the $\mathcal{N} = 4$ SYM theory [17, 18], we construct in section 3 an anisotropic spin chain with broken parity invariance. Though we will be interested in a fixed number of states per site, this number is kept arbitrary. We then construct its associated field theory and find the constraints for it to be supersymmetrizable. In section 4 we briefly review the q-deformation of the $\mathcal{N} = 4$ SYM, show that it matches the theory derived in section 3 and identify the sectors described by the corresponding spin chain. In section 5 we use the Bethe Ansatz to diagonalize the spin chain describing a 2-field sector of the theory. We also explicitly construct some low dimension eigen-operators of the dilatation operator. In section 7 we discuss possible extensions of this example.

2. A trivial construction

Let us consider an one-dimensional lattice integrable model with nearest neighbor interactions and periodic boundary conditions.[1] Its hamiltonian takes the usual form

$$H_J = \sum_{n=1}^{J} H_{n,n+1} \qquad J+1 \equiv 1\,, \tag{2.1}$$

where the index n labels the chain site and we assume that $H_{n,n+1}$ is independent of n. Each term in the sum (2.1) acts in the obvious way:

$$H_{n,n+1} : V_n \otimes V_{n+1} \to V_n \otimes V_{n+1} \tag{2.2}$$

and V_i is the space of states at the i-th lattice site.

This data suffices to construct a 4-dimensional field theory whose generator of scale transformations at 1-loop acts on a special class of scalar operators of bare dimension J as H_J, up to the addition of the identity operator. The construction is completely trivial. Consider the lagrangian

$$\mathcal{L} = \sum_i \mathrm{Tr}[\partial_\mu \phi_i \partial^\mu \bar{\phi}^i] + \sum_{i,j,k,l} \mathrm{Tr}[\phi_i \phi_j \bar{\phi}^k \bar{\phi}^l] \tilde{H}^{ji}_{kl} + \sum_{i,j} \mathrm{Tr}[\phi_j \bar{\phi}^i \phi_l \bar{\phi}^k] A^{jl}_{ik}\,, \tag{2.3}$$

where the fields ϕ are $n \times n$ matrices (with arbitrary n) and $\tilde{H}$ and A are, for the time being, arbitrary coefficients. A simple computation implies that, at the planar level, 1-loop scale transformations act on the holomorphic operators with fewer than n^2 fields as

$$D = \frac{\lambda}{4\pi} \sum_{i=1}^{J} \left[\, \tilde{H}^{m_i, m_{i+1}}_{n_i, n_{i+1}} + \alpha \delta^{m_i}_{n_i} \delta^{m_{i+1}}_{n_{i+1}} \right]. \tag{2.4}$$

In deriving this equation we used dimensional regularization and set all tadpole diagrams to zero. The first term arises from the intrinsic renormalization of the operator while the second term arises from the renormalization of the constituent fields (the lagrangian (2.3) leads to $\alpha = 0$, but we keep α unspecified for further convenience). Thus, the spin chain (2.1) diagonalizes the action of the dilatation operator on holomorphic operators in the theory (2.3) with

$$\tilde{H} = H_{n,n+1}\,, \tag{2.5}$$

while A remains arbitrary. In fact, we can insert an arbitrary multiplicative constant in the relation above. This coefficient will appear only as a multiplicative factor between the eigenvalues of the spin chain hamiltonian and the anomalous dimensions of field theory operators.

[1] The periodic boundary condition can easily be replaced with a twisted-periodic one. More general boundary conditions are also possible.

We have therefore shown that, given any integrable spin chain, there exists at least one field theory whose 1-loop dilatation operator acts on certain operators as the hamiltonian of the spin chain.

Under certain conditions it is possible to construct field theories such that the holomorphy constraint is absent. Indeed, if after lowering the upper indices the spin chain hamiltonian is cyclically symmetric up to the addition of the identity operator

$$\begin{aligned} H_{kl}^{ji} &\longrightarrow H_{kl;ji} \\ H_{kl;ji} + \beta\delta_{kj}\delta_{li} &= H_{lj;ik} + \beta\delta_{li}\delta_{jk} = H_{ji;kl} + \beta\delta_{jk}\delta_{il} = H_{ik;lj} + \beta\delta_{il}\delta_{kj} \end{aligned} \tag{2.6}$$

then the computation leading to (2.4) implies that the 1-loop dilatation operator of the theory with lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_i \mathrm{Tr}[\partial_\mu \phi^i \partial^\mu \phi^i] + \mathrm{Tr}[\phi^i \phi^j \phi^k \phi^l](H_{kl;ij} + \beta\delta_{kj}\delta_{li}) \tag{2.7}$$

is diagonalized by the eigenstates of the spin chain hamiltonian.

There are further theories, with similar properties, which differ from (2.3) and (2.7) by the addition of "flavor-blind" interactions. Such interactions (e.g. gauge interactions) contribute only to the coefficient α of the identity operator in (2.4) and thus do not modify the integrable properties of the 1-loop dilatation operator. From a field theory standpoint the different combinations of the terms described above lead to theories which are completely independent, though they can be occasionally deformed into each other by adjusting the coupling constants. It is however the case that the anomalous dimensions of operators in appropriate sectors are not independent, but they are determined by the eigenvalues of the transfer matrix of the same integrable spin chain.

2.1 A necessary condition for 1-loop integrability

As it is well known, a quantum integrable system is defined by its R-matrix which is a solution of the quantum Yang-Baxter equation (QYBE)

$$R_{12}{}^{b_1 b_2}_{a_1 a_2}(\lambda-\mu) R_{13}{}^{c_1 b_3}_{b_1 a_3}(\lambda) R_{23}{}^{c_2 c_3}_{b_2 b_3}(\mu) = R_{23}{}^{b_2 b_3}_{a_2 a_3}(\mu) R_{13}{}^{b_1 c_3}_{a_1 b_3}(\lambda) R_{12}{}^{c_1 c_2}_{b_1 b_2}(\lambda-\mu)\,. \tag{2.8}$$

The R-matrix determines the transfer matrix which can be used to reconstruct the hamiltonian of the system. More precisely, the logarithmic derivative of the transfer matrix evaluated at the value of the spectral parameter for which the R-matrix becomes the permutation operator is a spin chain hamiltonian with only nearest neighbor interactions (2.1) with

$$H_{n,n+1} \propto \frac{\partial R}{\partial \lambda}\Big|_{\lambda=\lambda_0} \qquad R(\lambda_0) = \mathcal{P}\,. \tag{2.9}$$

Consequently, the discussion in the previous section implies that the R-matrix of an integrable system is closely related to the coefficients of the 4-point interaction terms in the lagrangian of the associated field theory.

Finding solutions of the QYBE is in general a complicated task. The simple observation made above yields a simpler (though still algebraically rather involved) criterion for testing whether a given 4-point scalar interaction can be related to an integrable spin chain.

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The idea is very simple: by taking one derivative of the QYBE with respect to λ and evaluating the result at specific values for μ and λ we obtain a constraint on H. The most obvious choice, $\lambda = \lambda_0 = \mu$, leads to no useful constraints. A better choice turns out to be

$$\lambda \to \infty \qquad \mu \to \infty \qquad \lambda - \mu = \lambda_0 \,. \tag{2.10}$$

This limit is easy to analyze because for infinite spectral parameter the R-matrix becomes

$$R(\lambda) = \mathbb{1} \otimes \mathbb{1} + \frac{r}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \tag{2.11}$$

where r is a solution of the classical Yang-Baxter equation:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \,. \tag{2.12}$$

Similarly to R_{ij}, r_{ij} acts on $V_i \otimes V_j$ and the commutators are taken over repeated indices.

Interpreting $H_{12} \equiv H_{i_1 i_2}^{j_1 j_2}$ as an operator acting on $V_1 \otimes V_2$, simple algebra leads to the conclusion that, for H_{12} to be related to an integrable spin chain, it is necessary that

$$[\mathcal{P}H_{12}, r_{23}] + [\mathcal{P}H_{12}, r_{13}] = 0 \,, \tag{2.13}$$

where r is a solution of the classical Yang-Baxter equation and $(\mathcal{P}H)_{ab}^{cd} = H_{ba}^{cd}$. It is worth pointing out that the equation (2.13) is insensitive to the freedom of adding the identity operator in the relation between the spin chain and the 4-point coupling in the field theory lagrangian.

Thus, the conclusion is that for a given field theory with only 4-point interactions to have a chance to be related to an integrable spin chain, the equation (2.13) must be satisfied with H_{12} being the coefficient of the 4-point coupling and r being a solution of the classical Yang-Baxter equation. As an example let us briefly ilustrate this constraint with the holomorphic sector of $\mathcal{N} = 4$ SYM. In this case, H is a linear combination of the identity operator and permutation operator. The equation (2.13) is satisfied if r is also the permutation operator, which also satisfies the classical Yang-Baxter equation.

2.2 Symmetries

The lagrangians (2.3) and (2.7) have the same symmetries as the hamiltonian of the integrable spin chain they are based on. Some integrable spin chains have hamiltonians which are invariant under the action of some group G. In such situations the fields ϕ^i transform in some representation of the symmetry group. This is, however, the exception rather than the rule, since probably most integrable models have no nonabelian continuous bosonic symmetries. In section 3 we will construct such a model and analyze it in detail.

An interesting question is whether the lagrangian (2.3) can be embedded in a supersymmetric theory. It is easy to see that fermions can be added to (2.3) without spoiling the integrability of (2.4). Indeed, Lorentz invariance guarantees that at 1-loop level the only contribution of fermions to the scale transformation of bosonic scalar operators is through the wave function renormalization and thus affects only the precise value of the coefficient α in (2.4).

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Since the scalar fields in chiral multiplets are complex, it is natural to start with a field theory lagrangian of the type (2.3). Requiring that it is supersymmetrizable imposes certain factorization constraints on the spin chain hamiltonian (2.1), since the interaction term in (2.3) must be positive semidefinite. Thus, we find that $H_{n,n+1}$ must factorize as

$$H_{n,n+1}{}^{ji}_{kl} = \sum_{k}\sum_{M_k} C_{M_k\,kl}\bar{C}^{M_k\,ji}\,, \tag{2.14}$$

where the bar denotes complex conjugation and the ranges of the indices k and M_k are fixed by the solution of the factorization problem.

Even if H has this property, the generic situation is that the sum of the ranges of M_k exceeds the range of i and j. Then, the lagrangian (2.3) can be only a subsector of a larger theory, since it is necessary to add at least $\sum_k |M_k| - n$ extra fields. If this difference vanishes or equals unity the theory is special in the sense that, depending on C, it may not require extra fields for it to be supersymmetric (vanishing difference) or just one extra field which may be a gauge field (unit difference).

A full classification is cumbersome (and perhaps not very illuminating) without making further assumptions on H and C. We will therefore leave the general discussion and proceed to analyze in detail a specific example. In the next section we construct an integrable spin chain which is a deformation of one with U(K) invariance. In general, the resulting chain has no continuous nonabelian symmetries. For special values of the coupling constants the associated field theory can be embedded in a supersymmetric one. It will turn out that, if each site supports two states ($K = 2$), this chain describes the holomorphic operators in a 2-field sector of a q-deformation of $\mathcal{N} = 4$ SYM with arbitrary deformation parameter while for three states per site it describes all holomorphic operators of the q-deformation of $\mathcal{N} = 4$ SYM with a pure phase parameter.

3. An anisotropic spin chain with broken parity

As an example of the previous discussion, in the remainder of this note we will describe in the language of integrable spin chains (part of) the holomorphic sector of certain deformations of $\mathcal{N} = 4$ SYM theory which break its R-symmetry to a U(1)^3 subgroup.

Since the R-symmetry of $\mathcal{N} = 4$ SYM restricted to the holomorphic sector is SU(3) we should in principle look for R-matrices which act on the 3-dimensional representation of SU(3) but do not posses this symmetry. We will however keep the discussion general and construct an R-matrix acting on the fundamental representation of U(K). The advantage of doing this is that the result can be used to describe smaller sectors of the field theory. We will analyze in considerable detail a 2-field sector of the theory and find the spectrum of scaling dimensions of all chiral operators in this sector.

3.1 The R-matrix and the spin chain hamiltonian

Parameter-dependent R-matrices are a common feature in the theory of quantum groups. Besides the spectral parameter λ, R also depends on the quantum deformation parameter q. There also exist quantum groups with more than one deformation parameter. Their associated R-matrices will depend on these parameters and are clearly not invariant under the undeformed symmetry transformations. Even more general parameter-dependent R-matrices can be constructed.

Non-symmetric solutions of the quantum Yang-Baxter equation (2.8) are usually constructed on a case by case basis. Defining $(e^{ij})_{kl} = \delta^i_k \delta^j_l$ with $i, j, k, l = 1, \ldots, K$, it is easy though rather tedious to check that a solution of (2.8) is

$$\begin{aligned} R(\lambda) \,=\, & \left(q^{1+a\lambda} - q^{-1-b\lambda}\right) \sum_i e^{ii} \otimes e^{ii} + \left(q^{a\lambda} - q^{-b\lambda}\right) \sum_{i\neq j} e^{\alpha_{ij}} e^{ii} \otimes e^{jj} + \\ & + (q - q^{-1}) \left(q^{a\lambda} \sum_{i<j} e^{ij} \otimes e^{ji} + q^{-b\lambda} \sum_{i>j} e^{ij} \otimes e^{ji} \right), \end{aligned} \tag{3.1}$$

where λ is the spectral parameter while a, b, q and $\alpha_{ij} = -\alpha_{ji}$ are free parameters. Similar R-matrices appeared in [19, 20]. This R-matrix has all the properties described above and we will take it as the starting point of our construction.

Choosing the Lax operator to be the R-matrix, the monodromy matrix T, which is a solution of the equation

$$R_{12}(\lambda - \mu)(T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) R_{12}(\lambda - \mu) \tag{3.2}$$

and the transfer matrix are constructed out of (3.1) in the usual way:

$$T(\lambda)^{a_{n+1};j_1\ldots j_n}_{a_1\,;i_1\ldots i_n} = \sum_{a_2,\ldots a_n} R^{a_2 j_1}_{a_1 i_1} R^{a_3 j_2}_{a_2 i_2} \ldots R^{a_n j_{n-1}}_{a_{n-1} i_{n-1}} R^{a_{n+1} j_n}_{a_n i_n} \tag{3.3}$$

$$\tau(\lambda)^{j_1\ldots j_n}_{i_1\ldots i_n} = \sum_a T(\lambda)^{a;j_1\ldots j_n}_{a;i_1\ldots i_n}\,. \tag{3.4}$$

In the construction of the hamiltonian, an important value of the spectral parameter is the one for which the R-matrix becomes the permutation operator and the monodromy matrix becomes the shift operator. In our case, this value is $\lambda = 0$

$$R(0) = (q - q^{-1}) \sum_{i,j} e^{ij} \otimes e^{ji} = (q - q^{-1})\mathcal{P} \qquad \mathcal{P}^{k,l}_{i,j} = \delta^l_i \delta^k_j\,. \tag{3.5}$$

It is then easy to see that the transfer matrix evaluated at vanishing spectral parameter is the generator of shifts along the chain.

The hamiltonian of the spin chain is the logarithmic derivative of the transfer matrix evaluated (for the present case) at $\lambda = 0$. The equation (3.5) implies that it is a sum of nearest neighbor interaction terms, each of which is, up to normalization, the derivative of the R-matrix evaluated at $\lambda = 0$:

$$\begin{aligned} H_{n,n+1} \,=\, -b \ln q \Bigg[& \frac{1 - \Delta^2 q^2}{1 - q^2} \sum_i e^{ii}_n \otimes e^{ii}_{n+1} + \frac{q(1 - \Delta^2)}{1 - q^2} \sum_{i\neq j} e^{\alpha_{ij}} e^{ij}_n \otimes e^{ji}_{n+1} + \\ & + \Delta^2 \sum_{i<j} e^{ii}_n \otimes e^{jj}_{n+1} + \sum_{i>j} e^{ii}_n \otimes e^{jj}_{n+1} \Bigg], \end{aligned} \tag{3.6}$$

where we defined $\Delta^2 = -a/b$.

We are now ready to construct the field theory associated to this spin chain. In the construction above the range of the indices of the R-matrix was not fixed. In the following sections however we will be interested in specific cases in which i and j take two or three values.[2]

3.2 The associated field theory

To illustrate the construction in section 2 we will write down the associated field theory first for e^{ij} generating U(2) and then for arbitrary range for its indices.

In the first case e^{ij} can be expressed in terms of the usual Pauli matrices

$$e^{12} = \sigma^{+} \qquad e^{21} = \sigma^{-} \qquad e^{11} = \frac{1}{2}\left(\mathbb{1} + \sigma^{3}\right) \qquad e^{22} = \frac{1}{2}\left(\mathbb{1} - \sigma^{3}\right). \tag{3.7}$$

The resulting spin chain is a parity-violating[3] extension of the XXZ Heisenberg chain. The latter is recovered in the limit $q \to 1$, $\alpha \to 0$ and $a \to -b$ taken in this order at the level of the R-matrix. It is easy to see from (3.6) that this extension does not include the XYZ spin chain. As in the case of the XYZ chain, the natural U(2) symmetry of the system is broken by the presence of the various relative coefficients in (3.6).

From the construction in the previous section, a field theory associated to this spin chain has the following lagrangian:

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Bigg[\partial\phi_i\partial\bar{\phi}^i - F\, b \ln q \Bigg[& \frac{q(1-\Delta^2)}{1-q^2}\left(e^{\alpha}\phi_2\phi_1\bar{\phi}^2\bar{\phi}^1 + e^{-\alpha}\phi_1\phi_2\bar{\phi}^1\bar{\phi}^2\right) + \Delta^2\phi_2\phi_1\bar{\phi}^1\bar{\phi}^2 \\ & + \phi_1\phi_2\bar{\phi}^2\bar{\phi}^1 + \frac{1-\Delta^2 q^2}{1-q^2}\left(\phi_1\phi_1\bar{\phi}^1\bar{\phi}^1 + \phi_2\phi_2\bar{\phi}^2\bar{\phi}^2\right)\Bigg]\Bigg],\end{aligned} \tag{3.8}$$

where the coefficients identify the origin of each of the terms above in equation (3.6) and the coefficient F reflects the freedom to rescale the spin chain hamiltonian.

It is also not possible to extend (3.8) to a supersymmetric lagrangian without adding at least three more bosonic fields. As it is well known, a theory admits a supersymmetric extension if its potential is a sum of squares. There are many ways one can attempt to arrange the terms in (3.8) in this fashion. Furthermore, as pointed out in section 2, there also exists the freedom of adding terms which do not contribute to planar Feynman diagrams. Keeping this in mind, a useful way of writing (3.8) is

$$\begin{aligned}\mathcal{L} = \partial\phi_i\partial\bar{\phi}^i - F\,\ln q[k|\phi_1\phi_2 - W\,\phi_2\phi_1|^2 + C\,(\phi_1\bar{\phi}^1 + \phi_2\bar{\phi}^2)(\bar{\phi}^1\phi_1 + \bar{\phi}^2\phi_2) + \\ + A\phi_2\phi_1\bar{\phi}^2\bar{\phi}^1 + B\phi_1\phi_2\bar{\phi}^1\bar{\phi}^2],\end{aligned} \tag{3.9}$$

where

$$\begin{aligned} A &= kW + b\frac{q(1-\Delta^2)}{1-q^2}e^{\alpha} \qquad B = k\overline{W} + b\frac{q(1-\Delta^2)}{1-q^2}e^{-\alpha} \qquad C = b\,\frac{1-\Delta^2 q^2}{1-q^2} \\ k &= -b\frac{q^2\,(1-\Delta^2)}{1-q^2} \qquad |W|^2 = \frac{1}{q^2}\,. \end{aligned} \tag{3.10}$$

[2] Alternatively, e^{ij} generate U(2) and U(3), respectively.

[3] For $a \neq -b$ the interactions to the left are different from those to the right.

By adding terms contributing only to nonplanar (and perhaps self-energy) diagrams, the last term on the first line can be written as a perfect square of an imaginary object. Nevertheless, the lagrangian (3.9) cannot be supersymmetrized as long as $A \neq 0$ and $B \neq 0$. It may already be clear and we will see it in detail in section 4 that if A and B vanish, this theory is (part of) the q-deformation of the $\mathcal{N} = 4$ SYM.

In the general case, the interaction lagrangian of the field theory associated to the spin chain (3.6) is just

$$\begin{aligned}\frac{L_{\rm int}}{F \ln q} &= C \left(\sum_{i=1}^{n} \phi_i \bar\phi^i \right) \left(\sum_{i=1}^{n} \bar\phi^i \phi_i \right) + \sum_{i<j=1}^{n} k_{ij} |\phi_i \phi_j - W_{ij} \phi_j \phi_i|^2 + \\ &+ \sum_{i<j=1}^{n} A_{ij} \phi_j \phi_i \bar\phi_j \bar\phi_i + B_{ij} \phi_i \phi_j \bar\phi_i \bar\phi_i \,. \end{aligned} \tag{3.11}$$

It turns out that $k_{ij} = k$ and $|W_{ij}| = |W|$ for all indices $i, j = 1 \dots n$ while A_{ij}, B_{ij} and the phase of W_{ij} can be obtained from (3.10) by replacing α with α_{ij}. As before, if the terms on the second line vanish, this lagrangian can be supersymmetrized by adding an appropriate number of fields as well as the appropriate "nonplanar" terms.

It is important to note that the coefficient C in the equations (3.9) and (3.11) is not fixed. Indeed, we are free to add the identity operator to (3.6) without changing the eigenvectors and shifting the eigenvalues by its coefficient. This freedom will be important in the next section.

4. The q-deformation of $\mathcal{N} = 4$ SYM

In this section we will show that the supersymmetric limit of the lagrangians (3.9) and (3.11) describe, respectively, a 2-field sector of a q-deformation with generic deformation parameter of the $\mathcal{N} = 4$ SYM and the full holomorphic sector if the deformation parameter is a pure phase.

There is no *a priori* reason to expect that deformations of $\mathcal{N} = 4$ SYM should have integrable structures. Various sectors of $\mathcal{N} = 4$ SYM are described by certain integrable spin chains and some of them have deformations which preserve the integrability. Usually, such deformations are, however, rather difficult to find. The spin chain constructed in section 3 is an integrable deformation of the U(K) symmetric chain in the fundamental representation.

Let us begin by reviewing the q-deformation of the $\mathcal{N} = 4$ SYM. The superpotential is given by

$$W = \mathrm{Tr}[\Phi_1(\Phi_2\Phi_3 - w\Phi_3\Phi_2)] \,. \tag{4.1}$$

This deformation preserves $\mathcal{N} = 1$ supersymmetry and breaks the SU(4) R-symmetry to U(1)^3. Integrating out the auxiliary fields and restricting to the scalar sector the interaction terms are

$$\begin{aligned} V = &\ \mathrm{Tr}[\ |\phi_1\phi_2 - w\phi_2\phi_1|^2 + |\phi_2\phi_3 - w\phi_3\phi_2|^2 + |\phi_3\phi_1 - w\phi_1\phi_3|^2] + \\ &+ \mathrm{Tr}[\ ([\phi_1, \bar\phi^1] + [\phi_2, \bar\phi^2] + [\phi_3, \bar\phi^3])^2] \,, \end{aligned} \tag{4.2}$$

where the first three terms arise from integrating out the auxiliary fields in the chiral multiplets while the last term arises from integrating out the auxiliary fields in the vector multiplet.

A brief look at the appropriate Feynman diagrams shows that, as in the case of $\mathcal{N} = 4$ SYM ($w = 1$), operators built out of only two scalar fields form a sector closed under scale transformations. Furthermore, due to the large N limit, the terms contributing to the 1-loop scale transformations of these operators are obtained by setting one of the fields (say ϕ_3) to zero as well as keeping only the terms in which the conjugate fields appear next to each other

$$V = \mathrm{Tr}[\, |\phi_1\phi_2 - w\phi_2\phi_1|^2] - 2\,\mathrm{Tr}[\, (\phi_1\bar{\phi}^1 + \phi_2\bar{\phi}^2)(\bar{\phi}^1\phi_1 + \bar{\phi}^2\phi_2)]\,. \tag{4.3}$$

Comparing this expression with the supersymmetric limit of (3.9) we find that they are identical provided that we set

$$A = 0 \qquad B = 0 \qquad \Rightarrow \qquad W = \frac{e^{i\beta}}{q} \qquad \alpha = i\beta \qquad \beta \in \mathbb{R} \tag{4.4}$$

$$w = W \qquad F\,k\,\ln q = 1 \qquad \frac{C}{k} = -\frac{1-\Delta^2 q^2}{q^2(1-\Delta^2)} = -2\,. \tag{4.5}$$

The first four identifications are rather straightforward. The fifth one fixes Δ in terms of the absolute value of w. It is worth pointing out that all the free parameters appearing in the hamiltonian (3.6) were necessary to recover the details of (4.3).

As pointed out in the previous section, the relative coefficient between the two terms in (4.3) has no consequence in this match, since it is possible to adjust it by adding the identity operator to the spin chain hamiltonian. The holomorphic operators in the 2-field sector have the following structure

$$\mathcal{O} = \mathrm{Tr}\left[\prod_{i=1}^{n} \phi_1^{m_i}\phi_2^{p_i}\right] \tag{4.6}$$

with arbitrary n, m_i and p_i. We will construct the eigenstates of the generator of scale transformations in section 6. In the limit $w \to 1$ these operators reduce to those discussed in [21].

The three field sector can be matched with the lagrangian (3.11) for $n = 3$. In this case however there is a further constraint which arises because of the different order of fields in (3.11). Indeed, after adding terms which do not contribute in the planar limit, adjusting the coefficients F and C in (3.11) and taking the supersymmetric limit, the lagrangian associated to the 3-state spin chain becomes

$$\begin{aligned} V \;=\; & \mathrm{Tr}\left([\phi_1,\, \bar{\phi}^1] + [\phi_2,\, \bar{\phi}^2] + [\phi_3,\, \bar{\phi}^3]\right)^2 + \\ & + \;\mathrm{Tr}\,|\phi_1\phi_2 - W_{12}\phi_2\phi_1|^2 + |\phi_2\phi_3 - W_{23}\phi_3\phi_2|^2 + |\phi_1\phi_3 - W_{13}\phi_3\phi_1|^2\,, \end{aligned} \tag{4.7}$$

where $W_{ij} = q^{-1}e^{i\beta_{ij}}$ with $\alpha_{ij} = i\beta_{ij}$, similar to the two-field case. Thus, it matches the terms in (4.2) contributing to the 1-loop scale transformations of holomorphic operators

$$\mathcal{O} = \mathrm{Tr}\left[\prod_{i=1}^{n} \phi_1^{m_i}\phi_2^{p_i}\phi_3^{q_i}\right] \qquad (\forall)\, n,\ m_i,\ p_i \text{ and } q_i \tag{4.8}$$

only if $w = W_{12} = W_{23} = \overline{W}_{13}$ is a phase. This constraint amounts to taking the limit $q \to 1$ at the level of the hamiltonian (3.6). This is a rather singular limit, which should be taken after the supersymmmetry conditions $A_{ij} = B_{ij} = 0$ are imposed. It may be possible to relax the constraint that $|w| = 1$ by considering an even more general R-matrix. Finding the eigenvectors of the generator of scale transformations requires the diagonalization of the 3-state spin chain hamiltonian, which can be achieved through a nested Bethe Ansatz.

5. The Bethe ansatz in the 2-field sector

We will now diagonalize the hamiltonian (3.6) under the assumption that (e^{ij}) generate an SU(2) algebra. Since a systematic construction of the eigenstates relies on the details of the diagonalization procedure, we will describe it in some detail. Following [22], we will use the Algebraic Bethe Ansatz. This method also applies to spins with more than two states per site, as it was discussed in [20] for slightly different though very similar hamiltonians.

To fix the notation, if (e^{ij}) generate an SU(2) algebra in the fundamental representation, the monodromy matrix acts on a two-dimensional auxiliary space; its entries are operators which act on the quantum states

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} . \tag{5.1}$$

If the spin chain has J sites, the operators A, B, C and D act on the tensor product of J copies of the fundamental representation of SU(2), in spite of the fact that they are not SU(2) invariants. From (5.1) it follows that the transfer matrix is

$$\tau(\lambda) = \mathrm{Tr}_{\mathrm{aux}}\, T(\lambda) = A(\lambda) + D(\lambda) . \tag{5.2}$$

We will consider the following general R-matrix

$$R(\lambda-\mu) = \begin{pmatrix} f_u(\lambda,\mu) & 0 & 0 & 0 \\ 0 & g_u(\lambda,\mu) & h(\lambda,\mu) & 0 \\ 0 & k(\lambda,\mu) & g_l(\lambda,\mu) & 0 \\ 0 & 0 & 0 & f_l(\lambda,\mu) \end{pmatrix} , \tag{5.3}$$

with $f_u(\lambda,\mu) = f_u(\lambda-\mu)$, etc. In this section we will not need the specific forms of f, g, h and k. They are fixed by the QYBE, up to an overall multiplicative function of λ. Clearly, (3.1) is of this type.

The equation (3.2) implies certain commutation relations between the entries of the monodromy matrix evaluated at different values for the spectral parameter. For the present discussion, the most important ones are

$$[A(\lambda),\, A(\mu)] = 0 \qquad [B(\lambda),\, B(\mu)] = 0 \tag{5.4}$$

$$[C(\lambda),\, C(\mu)] = 0 \qquad [D(\lambda),\, D(\mu)] = 0 \tag{5.5}$$

$$A(\mu)B(\lambda) = \frac{f_u(\lambda,\mu)}{g_l(\lambda,\mu)}B(\lambda)A(\mu) - \frac{h(\lambda,\mu)}{g_l(\lambda,\mu)}B(\mu)A(\lambda) \tag{5.6}$$

$$D(\mu)B(\lambda) = \frac{f_l(\mu,\lambda)}{g_l(\mu,\lambda)}B(\lambda)D(\mu) - \frac{k(\mu,\lambda)}{g_l(\mu,\lambda)}B(\mu)D(\lambda) \tag{5.7}$$

which suggest that B and C can be interpreted as creation and annihilation operators. Thus, using a pseudo-vacuum state $|0\rangle$ satisfying

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle \qquad D(\lambda)|0\rangle = d(\lambda)|0\rangle \qquad C(\lambda)|0\rangle = 0 \tag{5.8}$$

the ansatz for the level N eigenstates of the transfer matrix is:

$$|\Psi_N(\{\lambda_i\})\rangle = \prod_{i=1}^{N} B(\lambda_i)|0\rangle\,. \tag{5.9}$$

Requiring that (5.9) is indeed an eigenstate of the transfer matrix leads to the Bethe equations, which constrain the arguments λ_i of the creation operators appearing in $|\Psi_N(\{\lambda_i\})\rangle$. Using (5.4)-(5.7) it is fairly easy to find that

$$A(\mu)|\Psi_(\{\lambda_i\})\rangle \;=\; \Lambda|\Psi_N(\{\lambda_i\})\rangle + \sum_{n=1}^{N}\Lambda_n B(\mu)\prod_{\substack{i=1\\ i\neq n}}^{N} B(\lambda_i)|0\rangle \tag{5.10}$$

$$D(\mu)|\Psi_N(\{\lambda_i\})\rangle \;=\; \tilde{\Lambda}|\Psi_N(\{\lambda_i\})\rangle + \sum_{n=1}^{N}\tilde{\Lambda}_n B(\mu)\prod_{\substack{i=1\\ i\neq n}}^{N} B(\lambda_i)|0\rangle \tag{5.11}$$

with the coefficient functions Λ given by

$$\Lambda \;=\; a(\mu)\prod_{i=1}^{N}\frac{f_u(\lambda_i,\mu)}{g_l(\lambda_i,\mu)} \qquad \tilde{\Lambda} = d(\mu)\prod_{i=1}^{N}\frac{f_l(\mu,\lambda_i)}{g_l(\mu,\lambda_i)} \tag{5.12}$$

$$\Lambda_n \;=\; -a(\lambda_n)\frac{h(\lambda_n,\mu)}{g_l(\lambda_n,\mu)}\prod_{\substack{i=1\\ i\neq n}}^{N}\frac{f_u(\lambda_i,\lambda_n)}{g_l(\lambda_i,\lambda_n)} \qquad \tilde{\Lambda}_n = -d(\lambda_n)\frac{k(\mu,\lambda_n)}{g_l(\mu,\lambda_n)}\prod_{\substack{i=1\\ i\neq n}}^{N}\frac{f_l(\lambda_n,\lambda_i)}{g_l(\lambda_n,\lambda_i)}\,. \tag{5.13}$$

In the equations above Λ and $\tilde{\Lambda}$ arise from using the first term in (5.6) and (5.7), respectively. The other (2^J-1) terms, coming from commuting A past the J factors of B, combine into Λ_n and $\tilde{\Lambda}_n$. To see that the expressions quoted above are correct we first note that (5.9) is completely symmetric in λ_i which implies that it is enough to compute Λ_1 and $\tilde{\Lambda}_1$. They arise from using the second term in (5.6) and (5.7) for pushing A and D past $B(\lambda_1)$ and the first term in those equations for the remaining commutators.

The equations (5.10) and (5.11) imply that the vectors (5.9) are eigenvectors of the transfer matrix if the terms depending on $B(\mu)$ cancel out. This condition leads to N equations which determine the arguments of the creation operators in terms of μ:

$$\Lambda_n + \tilde{\Lambda}_n = 0 \qquad \leftrightarrow \qquad \frac{a(\lambda_n)}{d(\lambda_n)} = \prod_{\substack{i=1\\ i\neq n}}^{N}\frac{f_l(\lambda_n,\lambda_i)}{f_u(\lambda_i,\lambda_n)}\frac{g_l(\lambda_i,\lambda_n)}{g_l(\lambda_n,\lambda_i)}\,, \tag{5.14}$$

where we used the identity

$$\frac{g_l(\mu,\lambda_n)h(\lambda_n,\mu)}{g_l(\lambda_n\mu)k(\mu,\lambda_n)} = -1 \tag{5.15}$$

which is a consequence of the QYBE.

These are the Bethe equations. Interestingly enough, these equations do not depend directly on the off-diagonal entries of the R-matrix. The fact that the R-matrix is not symmetric is reflected by the lack of a simple relation between $g_l(\lambda_i,\lambda_j)$ and $g_l(\lambda_j,\lambda_i)$ (which are "usually" the negative of each other).

There is an additional constraint on the arguments $\{\lambda_i\}$ of the creation operators B, which arises from the fact that we want to associate the eigenvectors of t with single-trace operators. The cyclicity of the trace translates into the constraint that (5.9) is invariant under shift operator $t(0)$:

$$a(0)\prod_{i=1}^{N}\frac{f_u(\lambda_i)}{g_l(\lambda_i)}+d(0)\prod_{i=1}^{N}\frac{f_l(-\lambda_i)}{g_l(-\lambda_i)}=1\,. \tag{5.16}$$

Out of the eigenvalues of the transfer matrix it is now trivial to construct the eigenvalues of the hamiltonian, by taking the appropriate logarithmic derivative with respect to μ and evaluating the result at $\mu=0$, as we did in (3.6) to construct the hamiltonian:

$$E(\{\lambda_i\})=\frac{\partial\ln(\Lambda+\tilde{\Lambda})}{\partial\mu}\Big|_{\mu=0}=\epsilon_0+\sum_{i=1}^{N}\epsilon_i\,. \tag{5.17}$$

In the equation above ϵ_0 arises from derivatives acting on $a(\mu)$ and $d(\mu)$, while ϵ_i contains the terms in which the derivative acts on the factor depending on λ_i in Λ and $\tilde{\Lambda}$. In some sense, ϵ_0 can be thought as some "vacuum energy", since it is proportional to the number of sites in the chain; its precise value depends on the normalization of the R-matrix. In the same spirit, ϵ_i can be thought as the contribution of a single creation operator to the energy.

It is now a simple exercise to apply these results to the case of the spin chain (3.6) and find the 1-loop anomalous dimensions of the holomorphic operators (4.6) in the theory (3.9). Since the supersymmetric case can be obtained from the non-supersymmetric one by a specific choice of parameters, we will keep the discussion general.

The diagonalization of the 3-state spin chain hamiltonian can be achieved through the nested Bethe Ansatz. The analysis is fairly similar to the 2-state case, though substantially more involved. For the spin chain corresponding to the Izergin-Korepin R-matrix, this was described in detail in [23], while spin chains similar to those in section 3 were analyzed in [20].

6. The eigenstates and eigenvalues

We will now specify the solution discussed in section 5 to the spin chain constructed in section 3. In this case the entries of the R-matrix are ($f(\lambda)\equiv f(\lambda,0)$, etc.):

$$f_u(\lambda) = f_l(\lambda)=q^{1+a\lambda}-q^{-1-b\lambda} \tag{6.1}$$

$$g_u(\lambda) = e^{\alpha}\left[q^{a\lambda}-q^{-b\lambda}\right]\qquad g_l(\lambda)=e^{-\alpha}\left[q^{a\lambda}-q^{-b\lambda}\right] \tag{6.2}$$

$$h(\lambda) = (q^2-1)q^{a\lambda-1}\qquad k(\lambda)=(q^2-1)q^{-b\lambda-1}\,. \tag{6.3}$$

The structure of the R-matrix implies that on each site the C operator acts as σ^+. Thus, the vacuum state $|0\rangle$ corresponds to the operator

$$O_0^J = \mathrm{Tr}[\phi_1^J] \equiv |\underbrace{\uparrow\uparrow\ldots\uparrow}_{J\ \text{times}}\rangle \tag{6.4}$$

where J is the number of sites in the chain. In the ket vector notation the up and down arrows correspond to ϕ^1 and ϕ^2, respectively. Thus, the arrows are cyclically-symmetric. This is important for correctly computing their normalization.

From this choice of the vacuum state it is easy to see that $a(\mu)$ and $d(\mu)$ in (5.8) are given by

$$a(\mu) = f_u(\mu)^J \qquad d(\mu) = g_l(\mu)^J\,. \tag{6.5}$$

Then, the vacuum and creation operator contribution to the eigenvalues of the transfer matrix are:

$$\epsilon_0 = J\,\frac{aq+b/q}{q-1/q}\ln q \qquad \epsilon_i = \frac{(a+b)(q-1/q)\ln q}{(1-q^{-(a+b)\lambda_i})(q^{1+(a+b)\lambda_i}-1/q)}\,. \tag{6.6}$$

Using (6.3) and (6.5) in (5.14) and taking the limit $q\to 1$ and $\alpha\to 0$ as well as shifting λ by a constant recovers the classic equations for the XXZ spin chain.[4]

The anomalous dimensions of the linear combinations of operators (4.6) corresponding to the eigenstates (5.9) are obtained from (6.6) by using the parameters F, a, b and q determined by (4.5) as well as adding the contribution of the wave function renormalization. It is not hard to see that this leads to a vanishing anomalous dimension for the ground state (6.4), as it should. It is interesting to note that this result holds even if supersymmetry is broken by having $A\neq 0$ or/and $B\neq 0$. This can be easily checked from field theory considerations by noticing that the susy-breaking terms in (3.9) do not contribute to the 1-loop scale transformation of the operator dual to the ground state (5.8).

The eigenvectors are constructed by acting with the creation operators $B(\lambda)$ on O_0^J. The expression of B for arbitrary J is rather complicated and perhaps not very enlightening. Roughly speaking, the action of a single creation operator creates a "spin wave". Consequently, the eigen-operators have a structure similar to that of the BMN operators, except that the weight of each term is more complicated than just a phase. Let us illustrate this by explicitly computing the eigen-operators with 2, 3 and 4 fields.

The entries of the monodromy matrix satisfy recurrence relations indexed by the number of sites in the chain. These relations follow from the expression of T in terms of the R-matrix. Denoting by A_N, B_N, C_N and D_N the entries of the monodromy matrix for a chain with N sites, the R-matrix is given by

$$R(\lambda) = P^+\otimes A_1 + P^-\otimes D_1 + \sigma^+\otimes B_1 + \sigma^-\otimes C_1 \tag{6.7}$$

with $A_1,\ B_1,\ C_1$ and D_1 given by (5.3):

$$A_1(\lambda) = f_u(\lambda)P^+ + g_u(\lambda)P^- \qquad B_1(\lambda) = h(\lambda)\sigma^- \tag{6.8}$$

$$C_1(\lambda) = k(\lambda)\sigma^+ \qquad D_1(\lambda) = g_l(\lambda)P^+ + f_l(\lambda)P^-\,. \tag{6.9}$$

[4] From the equation (6.6) it is clear that the order of limits needed to recover the XXZ spin chain is the one stated below equation (3.7): $q\to 1,\ \alpha\to 0$ and $a\to -b$

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It turns out that the recurrence relations following from (3.3) are partly diagonal, coupling only A_N with B_N and C_N with D_N. The relevant ones for our purpose are:

$$A_N = A_{N-1} \otimes A_1 + B_{N-1} \otimes C_1 \tag{6.10}$$
$$B_N = A_{N-1} \otimes B_1 + B_{N-1} \otimes D_1 \,. \tag{6.11}$$

It is fairly easy to see that, due to the cyclicity of the trace, the normalized eigen-operators with two and three fields are independent of the entries of the R-matrix. This is no longer the case for the operators with four fields which are more complex and (marginally) retain such a dependence:

$$\mathcal{O}_0^4 = |\uparrow\uparrow\uparrow\uparrow\rangle \qquad \mathcal{O}_1^4 = |\uparrow\uparrow\uparrow\downarrow\rangle \qquad \mathcal{O}_3^4 = |\uparrow\downarrow\downarrow\downarrow\rangle \qquad \mathcal{O}_4^4 = |\downarrow\downarrow\downarrow\downarrow\rangle \tag{6.12}$$
$$\mathcal{O}_2^4 = \frac{1}{N}\left[|\uparrow\uparrow\downarrow\downarrow\rangle + A|\uparrow\downarrow\uparrow\downarrow\rangle + A^2|\downarrow\uparrow\uparrow\downarrow\rangle\right] \tag{6.13}$$
$$N^2 = (1+A^2)^2 + A^2 \qquad \text{with} \qquad A = \frac{g_l(\lambda_2)}{f_u(\lambda_2)}\,.$$

Here N was computed in the planar limit. As stated in the beginning of this section, the structure of the operator $\mathcal{O}_2^4$ is similar to that of a BMN operator with two impurities, except that the phase weighting each term is replaced by powers of A. It is fairly easy to see that this structure persists for longer chains.

7. Discussion

The simple observations made in this note suggest a way to attempt to engineer a spin chain once a field theory is given. Generally speaking, once a bosonic field theory with only 4-point interactions is given, the interaction terms provide "boundary conditions" for the quantum Yang-Baxter equation. If a solution obeying the boundary conditions exists, finding it may still be a difficult algebraic problem. The boundary conditions provide, however, an ansatz for part (if not all) of the terms in the R-matrix. A useful strategy in finding a solution might be to interpret (if possible) the theory of interest as a deformation of a more symmetric theory.

The example described above admits a number of interesting generalizations. At the bosonic level and both in the 2-field and 3-field sectors it would be interesting to relax the holomorphy condition on the eigen-operators. In the spirit of section 2 and section 3, one would attempt to construct a multi-parameter deformation of the SO(4) and SO(6) spin chains and fix the free parameters by matching its hamiltonian with the field theory lagrangian expressed in terms of real fields.

Non-holomorphic operators in Wess-Zumino models have a simpler description. To see this we first notice that planar F-term interactions do not switch the positions of holomorphic and anti-homomorphic fields. Thus, given a set of operators with fixed ordering of chiral and anti-chiral fields, it seems likely that the corresponding spin chain is inhomogeneous in the sense that some sites carry different states than the other ones. Inclusion of gauge fields seems, unfortunately, to spoil this simple picture, since the D-term interactions do not preserve the ordering of holomorphic and anti-homomorphic fields.

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Acknowledgments

We would like to thank Iosif Bena, Nelia Mann, Mark Srednicki, Mark Spradlin and Anastasia Volovich for useful discussions and comments on the manuscript. This research was supported in part by the National Science Foundation under Grant No. PHY00-98395 as well as by the Department of Energy under Grant No. DE-FG02-91ER40618.

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